

Quantum Mechanics in Riemannian Space–Times.

I. The Canonical Approach

E.A.Tagirov

N.N.Bogoliubov Laboratory of Theoretical Physics

Joint Institute for Nuclear Research, Dubna, 141980, Russia

e–mail: tagirov@thsun1.jinr.dubna.su

Abstract

This paper is the first of two papers devoted to formulation of quantum mechanics of a particle in a normal geodesic frame of reference in the general Riemannian space–time. Here canonical quantization of geodesic motion in the 1+3–formalism is considered, the result of which will be compared in the subsequent paper II with that of the field–theoretical approach, see also gr-qc/9807030. The Schrödinger representation of quantum-mechanical kinematics and dynamics is presented in the general–covariant form and a physical interpretation of the state vectors and the position operators is discussed.

1 Introduction

The present paper is devoted to the canonical approach to quantum mechanics of the simplest quantum object, a neutral spinless point-like particle, in *the general Riemannian space-time* $V_{1,3}$ with intention to compare it in a subsequent paper with the version of the theory resulting in the field-theoretical approach. The canonical approach consists in quantization of the geodesical motion and the field-theoretical one starts with the quantum theory of the real scalar field in $V_{1,3}$ and, after a specification of the Fock space by a particle interpretation of states, restricts naturally defined second quantized operators of basic (primary) observables to the one-particle subspace. Since in the general $V_{1,3}$ there are no symmetry arguments, the particle interpretation can be based only on the concept of the particle as a localized object, which is well developed in the nonrelativistic quantum mechanics, see, e.g. [1], and leads as a result of formalization of measurement procedures to the Born's probabilistic interpretation of the state vector and position (coordinate) operators in the configuration representation.

Motivation of the interest to quantum mechanics in $V_{1,3}$ is very simple: consideration of quantum systems in general relativity including quantization of gravitation itself would be incomplete without study of consecutive generalization of the basic principles and notions of the standard quantum theory to the Riemannian space-time background. This idea is supported, on my opinion, by comparison of the two mentioned approaches which lead generally to different versions of quantum mechanics in $V_{1,3}$ which, of course, are consistent in the case of *the Minkowski space-time* $E_{1,3}$.

Quantization of geodesic motion in $V_{1,3}$ along the mentioned above standard lines has not been done apparently in view of a widespread opinion that actually it was considered mathematically rigorously by J.Śniatycki [2] in the framework of the geometrical quantization. However, it is well-known that the same classical theory results in different quantum counterparts depending on the procedure of quantization chosen. In [2] the formalism is chosen which considers all space-time coordinates as observables on an equal footing; I discuss it very briefly in Sec.2. The resulting representation space in this formalism is then $L^2(V_{1,3}, \mathbf{C})$ and thus localization should be considered on the space-time regions, whereas the standard quantum-mechanical formalism is based in fact on a 1+3-foliation of space-time and on the concept of a particle as a spatially localized object which is based on formalization of experimental procedures.

To this end, in Section 3 a 1+3-foliation based on the normal geodesic extension of a given Cauchy hypersurface Σ is introduced and the classical geodesic dynamics is represented in the reduced Hamilton form which is ready for canonical (i.e. $V_{1,3}$ is assumed to be an elementary manifold) quantization.

In Section 4 quantization of this system is done along the standard lines in the configura-

tion space. It should be noted that under quantization I mean here introduction of primary (basic) operators of observables of spatial position and momentum forming the so called quantum kinematics and construction of the Hamilton operator in terms of the primary operators, which provides with a quantum dynamics. Thus, the problem of construction of operators corresponding to any function on the phase space (the full quantization) related to the problem of ordering of the primary operators is only designated.

As a result, a structure is obtained which is very similar to the ordinary non-relativistic quantum mechanics but it is, of course, relativistic in the sense that c , the velocity of light, is any finite parameter. In Section 5 this structure is represented in a general covariant form which is useful, in particular, for further comparison with quantum mechanics resulting in the field-theoretical approach. In Section 6 a draft of physical interpretation of the obtained structure is exposed on the basis of relation between idealized "yes-no" measurement procedures forming a quantum logic and projection valued measures on a Hilbert space of states.

It should be noted at once that a heuristic (or naive) level of mathematical rigor is adopted in the paper and a majority of assertions are of the general situation, that is necessary mathematical conditions are supposed to be fulfilled. In particular, a proper treatment of unbounded operators, uniqueness and irreducibility of introduced representations of observables are not considered at all. I hope that it is plausible because my first aim is to outline possible changes in quantum mechanics related to introduction of the Riemannian metric of space-time. A necessary mathematical refinement can apparently be made along the known lines of the standard theory if the primary results and further development of them prove to be interesting.

Notation is standard for general relativity and, as a rule, in the simple index form. *The dot between differential operators denotes their operator product*, i.e. $\hat{A} \cdot \hat{B}$ means that $\hat{A} \cdot \hat{B} \psi(x) \equiv \hat{A}(\hat{B}\psi(x))$. Indices run the values: $\alpha, \beta, \dots = 0, 1, 2, 3$ and $i, j, \dots = 1, 2, 3$.

2 Preliminary Notes on Quantization

Quantization, as it usually referred to P.A.M.Dirac [3], is a linear map $Q : f \rightarrow \hat{f}$ of the Poisson algebra of real functions $f \in C^\infty(M_{2n})$ on a symplectic manifold (M_{2n}, Ω) , Ω being a symplectic form, to a set of operators acting in a pre-Hilbert space \mathcal{H} (*the representation space*), provided the following conditions are fulfilled:

- 1) $1 \rightarrow \hat{1}$, $\hat{1}$ being the unity operator in \mathcal{H} ;
- 2) $[f, g]_{\text{Poiss}} \stackrel{\text{def}}{=} \Omega^{ab} \partial_a f \partial_b g \rightarrow i\hbar^{-1}(\hat{f}\hat{g} - \hat{g}\hat{f}) \stackrel{\text{def}}{=} i\hbar^{-1}[\hat{f}, \hat{g}]$, $a, b, \dots = 1, \dots, 2n$;

3) $\hat{f} = (\hat{f})^\dagger$ on a dense subset of \mathcal{H} , where *the dagger denotes the Hermitean conjugation with respect to the inner product in \mathcal{H}* ;

4) a complete set of commuting operators $\hat{f}_1, \dots, \hat{f}_n$ exists, such that, if exists \hat{f} for which $[\hat{f}, \hat{f}_i] = 0$ for any i , then $\hat{f} = \hat{f}(\hat{f}_1, \dots, \hat{f}_n)$.

Unfortunately, these conditions together are apparently intrinsically inconsistent if one supposes arbitrary C^∞ functions f and g in the condition 2) or (and) an arbitrary M_{2n} in the condition 4). As concerns the condition 2), one finds an analysis of it and numerous references in [4]. The solution of the inconsistency proposed there is that only canonical coordinates $\{q, p\}$ in M_{2n} should satisfy 2) whereas commutators $[\hat{f}, \hat{g}]$ for other functions f, g should be determined in addition. The present paper is just restricted to consideration of the primary observables $\{q, p\}$.

As concerns 4), the case of dynamics of a point-like particle in $V_{1,3}$ admits a mathematically rigorous solution of the problem given in [2] where $M_8 \sim T^*V_{1,3}$ is taken the phase space. Canonical coordinates $\{p_{(\alpha)}, q^{(\beta)}\}$ on $T^*V_{1,3}$ are formed by a frame $\{p_{(\alpha)}\}$ on the typical fiber and by values $\{q^{(\beta)}\}$ of four independent functions $q^{(\beta)}(x)$, $x \in V_{1,3}$ on the base $V_{1,3}$. (I simplify the consideration assuming that $V_{1,3}$ can be covered by a single chart though essentially the geometrical quantization is a method for the situations when this is not the case.) Thus, $q^{(\beta)}(x)$ are classical observables defining a position in $V_{1,3}$, which should be mapped on operators by quantization, whereas the coordinates x^α are defined generally on a chart $U \subset V_{1,3}$ and provide the space-time with a primary manifold structure and are not quantized.

In the introduced notation the general-relativistic dynamics of a particle of the rest mass m on $V_{1,3}$ is determined by the constraint

$$m^2 c^2 = p_{(\alpha)} p_{(\beta)} \left(g^{\gamma\delta} \partial_\gamma q^{(\alpha)} \partial_\delta q^{(\beta)} \right) (x) \circ \pi, \quad (1)$$

where π is the projection in $T^*V_{1,3}$, i.e. any line in $T^*V_{1,3}$ satisfying this condition is a time-like geodesic.

The result of consecutive quantization of the described structure can be presented, denoting operators in \mathcal{H} by the "hat" mark, as follows [2]. Let $q(x) \in C^\infty(V_{1,3})$ and $K_\alpha(x)$ is a $C^\infty(V_{1,3})$ — one-form (covariant vector field) over $V_{1,3}$ (a section in $T^*V_{1,3}$). Then

$$Q : \{p_{(\alpha)}, q^{(\beta)}\} \rightarrow \{\hat{p}_{(\alpha)}, \hat{q}^{(\beta)}\}$$

is equivalent to the map

$$q(x) \rightarrow \hat{q}(x) \stackrel{def}{=} q(x) \cdot \mathbf{1}, \quad (2)$$

$$K_\alpha(x) \rightarrow \hat{p}_K(x) \stackrel{def}{=} -i\hbar(K^\alpha \nabla_\alpha + \frac{1}{2} \nabla_\alpha K^\alpha). \quad (3)$$

These operators act in $L^2(V_{1,3}; \mathbf{C}; \sqrt{-g}dx^0...dx^3)$, i.e. in the space of complex functions $\varphi(x)$ with the inner product

$$\langle \varphi_1, \varphi_2 \rangle = \int_{V_{1,3}} \bar{\varphi}_1 \varphi_2 \sqrt{-g} dx^0 ... dx^3. \quad (4)$$

The constraint Eq (1) is mapped by Q to the condition specifying in the representation space a subspace of functions satisfying the equation [2]

$$\begin{aligned} \square \varphi + \zeta R(x) \varphi + \left(\frac{mc}{\hbar} \right)^2 \varphi &= 0, \quad x \in V_{1,3} \\ \square &\stackrel{def}{=} g^{\alpha\beta} \nabla_\alpha \nabla_\beta, \end{aligned} \quad (5)$$

with $\zeta = 1/6$. This is a very satisfactory result because much earlier this value of ζ was found to provide with the correct quasiclassical behavior of one-particle states in the de Sitter-covariant quantization of a scalar field [5]. From there immediately follows the necessity of $\zeta = 1/6$ in any $V_{1,3}$ because there is no another scalar combination of components of the Riemann-Christoffel tensor with a dimensionless coefficient, which provides with the same term for the scalar field equation in the de Sitter space-time [6].

However, the structure so constructed is dissimilar to the standard quantum mechanics with the Born probabilistic interpretation in the configurational space, where the representation space is $L^2(E_3; \mathbf{C}; dx^1...dx^3)$ and time is the evolution parameter, not an observable. If one would consider $\varphi(x)$ as a wave function of a quantum object, this object is not stable because $\varphi(x)$ have to vanish in the time-like directions as well as in the space-like ones. For example, any superpositions of the usual positive or negative frequency plane wave solutions of Eq.(5) in $E_{1,3}$ has an infinite norm according to Eq. (4). A more detailed consideration of this question can be found in [7].

It is clear that the roots of the divergence are in the choice of $T^*V_{1,3}$ as initial M_{2n} and in the symmetric treatment of space and time coordinates. Actually, a moment of time, contrary to a space position, is not a property of a particle in the ordinary sense. Quantization of the geodesical dynamics after some sort of the 1+3-foliation of $V_{1,3}$ by three-dimensional configuration spaces enumerated by a time-like evolution parameter would better correspond to this purpose.

3 Classical Hamilton Theory of Geodesics

A time-like geodesic in $V_{1,3}$ is a line $x^\alpha = x^\alpha(s)$ providing with an extremum of the action integral between given points $x_1 = x(s_1)$, $x_2 = x(s_2)$

$$W = -mc \int_{s_1}^{s_2} ds \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}} \stackrel{def}{=} \int_{s_1}^{s_2} L' ds \quad (6)$$

The Lagrangian L' is singular in the sense that the components of four-momentum

$$p_\gamma(s) \stackrel{def}{=} \frac{dL'}{d(dx^\alpha/ds)} = -mc \left(g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \right)^{-1/2} g_{\gamma\delta} \frac{dx^\delta}{ds} \quad (7)$$

are not independent but satisfy the constraint

$$g^{\alpha\beta} p_\alpha p_\beta = m^2 c^2, \quad (8)$$

which is just Eq.(1) on a geodesic line $\{p_{(\alpha)} = p_\alpha(s), q^{(\alpha)} = x^\alpha(s)\}$. It is a consequence of invariance of the action with respect to arbitrary "gauge" transformation $s' = f(s)$. Owing to Eq.(8), the Hamiltonian

$$H' = p_\alpha \frac{dx^\alpha}{ds} - L' \quad (9)$$

vanishes: $H' = 0$.

There exist methods of gaugeless quantization and they apparently are equivalent to "general-relativistic" quantization described in Section 2. However just due to the arguments at the end of the section, I reduce the Hamilton formalism to the 1+3-foliated form fixing a Cauchy hypersurface $\Sigma = \{\Sigma(x) = const, x \in V_{1,3}\}$, introducing normal Gaussian coordinates $\{t, q^i\}$ based on Σ , in which

$$ds^2 = c^2 dt^2 - \omega_{ij}(t, q) dq^i dq^j \quad (10)$$

and choosing $s = t$ in W :

$$W = -mc \int_{t_1}^{t_2} dt \sqrt{c^2 - \omega_{ij}(t, q) \dot{q}^i \dot{q}^j}, = \int_{t_1}^{t_2} dt L, \quad (11)$$

where $\dot{q}^i \stackrel{def}{=} dq^i/dt$. I suppose also that $t = 0$ on the chosen Σ . Then three independent space momenta are

$$p_i \stackrel{def}{=} \frac{\partial L'}{\partial \dot{q}^i} = \frac{mc \omega_{ij}(t, q) \dot{q}^j}{\sqrt{c^2 - \omega_{ij}(t, q) \dot{q}^i \dot{q}^j}}. \quad (12)$$

and the Hamiltonian is

$$H \stackrel{def}{=} p_i \dot{q}^i - L = mc \sqrt{c^2 - \omega_{ij}(t, q^i) \dot{q}^i \dot{q}^j}. \quad (13)$$

Now, the Hamiltonian form of geodesic equation is

$$\dot{q}^i = [H, q^i]_{\text{Poiss}}, \quad \dot{p}_i = [H, p_i]_{\text{Poiss}}. \quad (14)$$

The spatial coordinates $q^i(t)$ and momenta $p_i(t)$ are the basic classical canonically conjugate (i.e. conjugate with respect to the canonical symplectic form Ω) observables of a particle and other classical observables are real functions of them.

4 Canonical Quantization of Geodesic Motion in $V_{1,3}$

Now, let a representation space for quantum operators will be $L^2(\Sigma; \mathbf{C}; dq^1 dq^2 dq^3)$, i.e. the space of the square-integrable over Σ and sufficiently smooth complex functions $\phi(q)$. A Hilbert structure is defined on it by the inner product

$$\langle \phi_1, \phi_2 \rangle_\Sigma \stackrel{def}{=} \int_\Sigma \bar{\phi}_1 \phi_2 dq^1 dq^2 dq^3, \quad \phi_1, \phi_2 \in L^2(\Sigma; \mathbf{C}; dq^1 dq^2 dq^3). \quad (15)$$

Introduce operators \tilde{q}^i, \tilde{p}_j , as follows

$$\tilde{q}^i \stackrel{def}{=} q^i \cdot \hat{\mathbf{1}}, \quad \tilde{p}_j \stackrel{def}{=} -i\hbar \frac{\partial}{\partial q^j} \quad (16)$$

which satisfy the Heisenberg commutation relations and are Hermitean (symmetrical) with respect to the inner product Eq.(15). Thus, they, together with the unity operator $\hat{\mathbf{1}}$, satisfy the conditions 1) — 3) of quantization. Then, take them as operator Cauchy data on Σ for the map Q of the Hamilton equations Eq.(14):

$$i\hbar \frac{d}{dt} \tilde{q}^i(t) = [\tilde{H}(t), \tilde{q}^i(t)], \quad i\hbar \frac{d}{dt} \tilde{p}_i(t) = [\tilde{H}(t), \tilde{p}_i(t)] \quad (17)$$

which are obviously the Heisenberg equations. If $\tilde{H}(t) \equiv \tilde{H}(t, \tilde{q}(t), \tilde{p}(t))$ is a Hermitean operator, then $\tilde{q}^i(t), \tilde{p}_j(t)$ are also Hermitean operators owing to hermiticity of the Cauchy data. Simultaneous hermiticity of $\tilde{q}^i(t), \tilde{p}_j(t)$, and $\tilde{H}(t)$ is provided by the following symmetric substitution of the basic operators $\tilde{q}^i(t), \tilde{p}_j(t)$ into the classical Hamiltonian $H(t)$:

$$\tilde{H}(t) = mc\sqrt{c^2 - \tilde{p}_i(t) \omega^{ij}(t, \tilde{q}(t)) \tilde{p}_j(t)}, \quad (18)$$

However, there is an infinite set of other possibilities of Hermitean ordering of the cofactors $\tilde{p}_i(t), \tilde{p}_j(t)$ and $\omega^{ij}(t, \tilde{q}(t))$. This is the well-known ambiguity of ordering in construction of a Hamilton operator from a classical Hamiltonian. For definiteness, I take for further consideration Eq.(18) as the Hamilton operator; apparently this is the simplest choice.

Thus, the conditions 1) — 3) of quantization are satisfied with regard to the basic observables and the Hamiltonian. As concerns condition 4), its consideration demands a formulation on the level of geometric quantization that seems, in the 1+3-formalism, to be much more complicated than in the four-dimensional formalism mentioned in the Section.2 since one should consider different charts on $V_{1,3}$ and Σ simultaneously. One should also keep in mind the known problems of mathematical justification of the use of unbounded operators, but, on the heuristic level accepted here, it is natural to suppose that it can be done in a way similar to the standard theory.

Now, let us introduce the Schrödinger wave functions $\phi_{\text{Sch}}(t, q)$ in the usual way and, in addition, transform them as follows:

$$\phi(q) \rightarrow \phi_{\text{Sch}}(t, q) = \omega^{1/4} \exp\left(-i \frac{mc^2}{\hbar} t\right) \psi(t, q), \quad \omega \stackrel{\text{def}}{=} \det\|\omega^{ij}(t, q)\|. \quad (19)$$

The Heisenberg equations Eq.(17) generate the following Schrödinger equation for $\psi(t, q)$ (the index \mathbf{c} will mean further "canonical")

$$i\hbar \left(\frac{\partial}{\partial t} + \omega^{-1/4} \frac{\partial \omega^{1/4}}{\partial t} \right) \psi = \hat{H}^{\mathbf{c}}(t, q, \hat{p}^{\mathbf{c}}) \psi. \quad (20)$$

where

$$\hat{H}^{\mathbf{c}}(t, q, \hat{p}^{\mathbf{c}}) = mc^2 \left(\sqrt{1 + \frac{2\hat{H}_0^{\mathbf{c}}}{mc^2}} - 1 \right) \quad (21)$$

$$\hat{H}_0^{\mathbf{c}} = \hat{H}_0^{\mathbf{c}}(t, q, \hat{p}^{\mathbf{c}}) \stackrel{\text{def}}{=} \frac{1}{m} \hat{p}_i^{\mathbf{c}} \omega^{ij}(t, q) \hat{p}_j^{\mathbf{c}} \quad (22)$$

and

$$\hat{p}_i^{\mathbf{c}} \stackrel{\text{def}}{=} -i\hbar \left(\frac{\partial}{\partial q^i} + \omega^{-1/4} \frac{\partial \omega^{1/4}}{\partial q^i} \right). \quad (23)$$

Note that $\hat{H}^{\mathbf{c}}$ is written as depending on the c -numbers q^i because, according to Eq.(16), in the Schrödinger picture, a coordinate operator is $\hat{q}_{\mathbf{c}}^i = \hat{q}^i|_{\Sigma} \equiv q^i \cdot \hat{\mathbf{1}}$. It is easily seen also that

$$\hat{H}_0^{\mathbf{c}} = -\frac{\hbar^2}{2m} \Delta_t + V \quad (24)$$

where Δ_t is the Laplacian on hypersurfaces

$$t = \{ \{t, q\} \in V_{1,3}; \ t = \text{const}; \ \Sigma \sim t = 0 \}$$

enumerated by a value of the evolution parameter t ; I denote them by the same letter t since it does not cause a confusion in a context.

$$V = V(t, q) = \frac{\hbar^2}{32m} \omega^{ij} \partial_i \log \omega \partial_j \log \omega \quad (25)$$

is a so called *quantum potential*. Another choice of ordering in transition from H to $\check{H}_0^{\mathbf{c}}$ would lead to a different $V(t, q)$ but the general structure of the Hamilton operator retains the form of Eq.(24) for any ordering. However, the question of different quantum potentials deserves a further consideration.

Denote Ψ the space of sufficiently smooth solutions of the Schrödinger equation(20). Then, from (15) one has for $\psi_1, \psi_2 \in \Psi$ determined by given initial $\phi_1, \phi_2 \in L^2(\Sigma; \mathbf{C}; dq^1 dq^2 dq^3)$

$$\langle \phi_1, \phi_2 \rangle_{\Sigma} = \int_{\Sigma} \bar{\psi}_1(t, q) \psi_2(t, q) \sqrt{\omega(t, q)} dq^1 dq^2 dq^3 \stackrel{\text{def}}{=} (\psi_1, \psi_2)_t. \quad (26)$$

Thus, $(\psi_1, \psi_2)_t$, which formally depends on t , actually does not. It can be considered as a scalar product in Ψ and written as

$$(\psi_1, \psi_2)_t = \int_t \bar{\psi}_1 \psi_2 d\sigma_t \quad (27)$$

where

$$d\sigma_t = \sqrt{\omega(t, q)} dq^1 dq^2 dq^3 = N_\alpha(x) d\sigma_t^\alpha(x)$$

is *the invariant volume element* of the hypersurface t , $N^\alpha(x)$ is *the normal geodesic extension* of the unit normal to Σ , which is also a unit normal to t , containing the point x , and $d\sigma_t^\alpha(x)$ is the normal volume element of t .

However, if \hat{O} is an operator which do not commute with the Hamilton operator, then $\hat{O}\psi \notin \Psi$ but as a function on a given hypersurface t it belongs to $L^2(t; \mathbf{C}; d\sigma_t)$ with the same scalar product $(\psi_1, \psi_2)_t$ the values of which depend on t if $\psi_1, \psi_2 \notin \Psi$. Thus we have an one-parametric family of L^2 -spaces, each over a layer of the introduced 1+3-foliation.

The (unbounded) operators $\hat{q}_\mathbf{c}$, $\hat{p}^\mathbf{c}$ and, owing to them, the operator $\hat{H}^\mathbf{c}$, are *Hermitean* in each $L^2(t; \mathbf{C}; d\sigma_t)$, that is

$$(\psi_1, \hat{O}\psi_2)_t = (\hat{O}\psi_1, \psi_2)_t, \quad (28)$$

on a dense subset of $L^2(t; \mathbf{C}; d\sigma_t)$.

5 Covariantization

Recall now that t, q^i , are in fact functions of the initially introduced general coordinates $\{x^\alpha\}$ so that $t = t(x)$, $q^i = q^{(i)}(x)$ and

$$\partial_\alpha t(x) \partial^\alpha q^{(i)}(x) = 0, \quad \text{rank} \|\partial_\alpha q^{(i)}(x)\| = 3; \quad (29)$$

Enclosing the index i in the brackets I denote that $q^{(i)}(x)$ as well as $t(x)$ are *given functions*, *i.e. scalar functions with respect to transformation of the coordinates x^α* . Thus, cf. Eq.(2),

$$\hat{q}_\mathbf{c}^i = \hat{q}_\mathbf{c}^{(i)}(x) = q^{(i)}(x) \cdot \hat{\mathbf{1}}. \quad (30)$$

Introduce, further, three vector fields $K_{(i)}^\alpha(x)$ on $V_{1,3}$ defined by conditions

$$\partial_\alpha t(x) K_{(i)}^\alpha(x) = 0, \quad K_{(i)}^\alpha(x) \partial_\alpha q^{(j)}(x) = \delta_{(i)}^{(j)} \quad (31)$$

which mean that $K_{(i)}^\alpha(x)$ lie on the hypersurface $t_\Sigma = \{x \in V_{1,3}; t(x) = \text{const}\}$ which contains the point x and is conjugate to the coordinate q^i on t_Σ . Then the operator $\hat{p}_{(i)}$

can be represented as (cf. Eq.(3))

$$\begin{aligned}\hat{p}_{(i)}^{\mathbf{c}} = \hat{p}_{(i)}^{\mathbf{c}}(x) &= -i\hbar(K_{(i)}^{\alpha}\nabla_{\alpha} + \frac{1}{2}\nabla_{\alpha}K_{(i)}^{\alpha}) \equiv \frac{i\hbar}{2}(K_{(i)}^{\alpha} \cdot \nabla_{\alpha} + \nabla_{\alpha} \cdot K_{(i)}^{\alpha}) \\ &\equiv -i\hbar(K_{(i)}^{\alpha}D_{\alpha} + \frac{1}{2}D_{\alpha}K_{(i)}^{\alpha}) \equiv \frac{i\hbar}{2}(K_{(i)}^{\alpha} \cdot D_{\alpha} + D_{\alpha} \cdot K_{(i)}^{\alpha})\end{aligned}\quad (32)$$

where

$$D_{\alpha} \stackrel{def}{=} h_{\alpha}^{\beta}(x)\nabla_{\beta} \quad \text{and} \quad h_{\alpha}^{\beta}(x) \stackrel{def}{=} \delta_{\alpha}^{\beta} - c^{-2}\partial_{\alpha}t(x)\partial^{\beta}t(x) \quad (33)$$

is the projection tensor on t . Thus, D_{α} is the projection of the covariant derivative on t and any observable $\hat{F}(t, \hat{q}_{\mathbf{c}}, \hat{p}^{\mathbf{c}}) \equiv \hat{F}(t, q, \hat{p}^{\mathbf{c}})$ is an element of the closure of Hermitean differential operators of with coefficients depending on x , which contain only the derivatives D_{α} and are Hermitean in $L^2(t; \mathbf{C}; d\sigma_t)$.

Further, let \mathbf{Q} and \mathbf{K} are the sets of sufficiently smooth functions $q(x)$ and vector fields $K^{\alpha}(x)$ satisfying the first of conditions in Eq. (29) and Eq.(31) correspondingly. Since any $q(x) \in \mathbf{Q}$ and $K^{\alpha}(x) \in \mathbf{K}$ can be expressed correspondingly as a function of given space position functions $q^{(i)}(x)$ and as a linear combination of space basis functions $K^{\alpha}(x)$, one can generalize the map

$$Q_{\mathbf{c}} : \quad q^{(i)} \rightarrow \hat{q}_{\mathbf{c}}^{(i)}(x), \quad p_{(j)} \rightarrow \hat{p}_{(j)}^{\mathbf{c}}(x)$$

to the map, cf. Eq.(2), (3),

$$Q : \quad \mathbf{Q} \ni q(x) \rightarrow \hat{q}_{\mathbf{c}}(x) = q(x)\hat{\mathbf{1}}, \quad (34)$$

$$\begin{aligned}\mathbf{K} \ni K^{\alpha}(x) \rightarrow \hat{p}_{\mathbf{K}}^{\mathbf{c}}(x) &= -i\hbar(K^{\alpha}\nabla_{\alpha} + \frac{1}{2}\nabla_{\alpha}K^{\alpha}) \equiv \frac{i\hbar}{2}(K^{\alpha} \cdot \nabla_{\alpha} + \nabla_{\alpha} \cdot K^{\alpha}) \\ &\equiv -i\hbar(K^{\alpha}D_{\alpha} + \frac{1}{2}D_{\alpha}K^{\alpha}) \equiv \frac{i\hbar}{2}(K^{\alpha} \cdot D_{\alpha} + D_{\alpha} \cdot K^{\alpha})\end{aligned}\quad (35)$$

These operators satisfy formally the generalized Heisenberg commutation relations:

$$[\hat{q}_{\mathbf{c}}, \hat{q}'_{\mathbf{c}}] = 0, \quad q(x), q'(x) \in \mathbf{Q} \quad (36)$$

$$[\hat{p}_{\mathbf{K}}^{\mathbf{c}}, \hat{p}_L^{\mathbf{c}}] = -i\hbar\hat{p}_{[K, L]_{\text{Lie}}}, \quad K^{\alpha}(x), L^{\alpha}(x) \in \mathbf{K}, \quad (37)$$

$$[\hat{q}_{\mathbf{c}}, \hat{p}_{\mathbf{K}}^{\mathbf{c}}] = i\hbar K^{\alpha}\partial_{\alpha}q, \quad (38)$$

where $[K, L]_{\text{Lie}}$ is the Lie derivative of a vector field $L^{\alpha}(x)$ along $K^{\alpha}(x)$, that is $[K, L]_{\text{Lie}}^{\alpha}(x) = K^{\beta}\nabla_{\beta}L^{\alpha} - L^{\beta}\nabla_{\beta}K^{\alpha}$. These relations were obtained in [8, p.146] from a system of imprimitivity for a symmetry group G acting on the configuration space (i.e. $t(x) = \text{const}$ in our case) so that \mathbf{K} is the Lie algebra of G . We came to these relations for the Lie algebra of smooth vector fields on t with no assumption on existence of a symmetry but the mathematical rigor of our consideration is essentially less than that of [8].

Thus, we have constructed covariantly what is often called the kinematical structure of quantum mechanics of a particle and proceed further with its dynamics.

The Schrödinger equation Eq.(20) can be rewritten now in the scalar (general covariant) form as follows:

$$i\hbar T(x) \psi(x) = \hat{H}^{\mathbf{c}}(x, \hat{p}^{\mathbf{c}}) \psi(x) \quad (39)$$

$$T \stackrel{def}{=} c^2 \left(\partial^\alpha t(x) \partial_\alpha + \frac{1}{2} \square t(x) \right) \equiv \frac{c^2}{2} (\partial^\alpha t(x) \cdot \partial_\alpha + \partial_\alpha \cdot \partial^\alpha t(x)) \quad (40)$$

where $\hat{H}^{\mathbf{c}}(x, \hat{p}^{\mathbf{c}})$ has the same form of the right-hand side of Eq.(21) but with

$$\hat{H}_0^{\mathbf{c}} = \hat{H}_0(x, \hat{p}^{\mathbf{c}}) \stackrel{def}{=} \hat{p}_i^{\mathbf{c}} \partial_\alpha q^{(i)} \partial^\alpha q^{(j)} \hat{p}_j^{\mathbf{c}}, \quad (41)$$

This nonrelativistic Hamilton operator has again the form of Eq.(24) with the quantum potential

$$V = V(x) = \frac{\hbar^2}{8m} \partial_\alpha K_{(i)}^\alpha \partial_\gamma q^{(i)} \partial^\gamma q^{(j)} \partial_\beta K_{(i)}^\beta \quad (42)$$

Thus, by means of the quantum potential, the canonical Hamilton operator essentially depends also on the choice of the independent functions $q^{(i)}(x)$, which are in fact *the normal geodesic extensions* of the functions on the initial Cauchy hypersurface Σ i.e. their values are constant along the geodesics which are normal to Σ . It should be remarked that generally the normal geodesic extension can be valid only in a limited normal geodesic distance from Σ because, for any Σ except the hyperplanes in $E_{1,3}$, the normal geodesics have intersections (focal points). Can this complication be treated by the Maslov-Fedoriuk methods [9], is an interesting but open question for me.

6 A Draft of Interpretation

In nonrelativistic quantum mechanics, relation between physical experiments over a quantum system and a the mathematical structure of quantum mechanics is based generally on a correspondence between an idealized "yes-no" experimental device ("proposition") and a σ -homomorphism of Borel subsets Δ of a σ -algebra \mathcal{S} to a lattice of subspaces in a Hilbert space \mathcal{H} . An excellent presentation of this topic is given by J.M.Jauch in the monograph [1] to which I try to follow.

In our case of description of a particle in the configuration space, a Borel set Δ_t is generated on the hypersurfaces t by intersections of the latter with the normal geodesics that issue from a Borel set Δ_Σ on the initial hypersurface Σ . Thus, Δ_t as a function of the parameter t is the normal geodesic extension of Δ_Σ . One may also imagine it as a Borel

subset over the set of all geodesics normal to Σ , but with account of the remark at the end of the preceding section and topological complications.

The procedure of detection of a particle in the Borel sets $\Delta_t \subset \{t : t = \text{const}\}$ is associated usually with a lattice in $\mathcal{H} \sim L^2(t; \mathbf{C}; d\sigma_t)$ generated by projection operators $\hat{E}(\Delta_t)$ defined as follows [1, p. 1] :

$$(\hat{E}(\Delta_t)\psi)(x) = \chi_{\Delta_t}\psi(x), \quad \psi \in L^2(t; \mathbf{C}; d\sigma_t) \quad (43)$$

where $\chi_{\Delta_t}(x)$ is the characteristic function of Δ_t . Thus one has a projection valued measure on

$$\Delta_t \rightarrow \hat{E}(\Delta_t) \quad (44)$$

which determines for each $\psi \in L^2(t; \mathbf{C}; d\sigma_t)$ a numerical measure

$$(\psi, \hat{E}(\Delta_t)\psi)_t = \int_{\Delta_t} d\sigma_t \bar{\psi} \psi \quad (45)$$

which can be interpreted, after normalization of ψ to unity, as a probability to find a particle in Δ_t in the state determined by ψ . Of course, this is a refined formulation and generalization to $V_{1,3}$ of the well-known Born probabilistic interpretation of function $\psi(x)$. It is important to realize that we owe by this interpretation to L^2 -structure of \mathcal{H} which is in fact a postulate as well as the canonical commutative realization (43) of system of projections $\hat{E}(\Delta_t)$.

However, providing with the interpretation of vectors of the representation space, the projection valued measure (44) on t defines nothing similar to $\hat{q}^{(i)}$ corresponding to a position observable in the phase space of a particle. In the case of Cartesian coordinates X^i in $\Sigma \sim E_3$, the operators corresponding to a mesurment of them are defined [1] through a system of projections $\hat{E}(\Delta^i(\lambda_i))$ associated with the class of subsets of E_3

$$\Delta^i(\lambda_i) \stackrel{\text{def}}{=} \{X^1, X^2, X^3 \mid X^i < \lambda_i\}, \quad -\infty < \lambda_i < \infty, \quad i = 1, 2, 3. \quad (46)$$

This class generates a σ -algebra and thus a Borel structure on each line which is parallel to the $X^{(i)}$ -axis.

In our case of a general Σ in $V_{1,3}$, having taken three functions $q^{(1)}(x)$, $q^{(2)}(x)$, $q^{(3)}(x)$, satisfying conditions (29) one may apparently generate a σ -algebra $\mathcal{S}^{(i)}$ on each integral curve of the vector field of the $K_{(i)}^\alpha$ in Σ and in its normal geodesic extension to t by the following class of semiopen subsets on Σ :

$$\Delta^{(i)}(\lambda_i) \stackrel{\text{def}}{=} \{q^1, q^2, q^3 \mid q^{(i)} < \lambda_i\}, \quad q_m^{(i)} \leq \lambda_i < q_M^{(i)}, \quad i = 1, 2, 3. \quad (47)$$

where $q_m^{(i)}$ and $q_M^{(i)}$ are the minimal and maximal possible values of $q^{(i)}$. So it is taken into account that, in principle, the coordinate curves may have one or two endpoints or be

closed but one should have in mind possible complications in the curvilinear case such as intersections of different coordinate lines of the same coordinate. A simple model example is the intersection of coordinate lines of ϕ in the center of the spherical coordinates r, ϕ on E_2 .

Further, let for definiteness $i = 1$ and let $\Delta^{(1)}$ is a Borel set of $\mathcal{S}^{(1)}$. One can introduce a projection valued measure

$$\mathcal{S}^{(1)} \supset \Delta^{(1)} \rightarrow \hat{E}(\Delta^{(1)}) \quad (48)$$

on each $q^{(1)}$ -coordinate curve. Let us assume, in addition, that it is possible to introduce projections $\hat{E}(\Delta^{(1)}(\lambda_1))$ on the system of subsets Eq.(47), which satisfy the following conditions (the spectral property):

$$\hat{E}(\Delta^{(1)}(\tilde{\lambda}_1)) \leq \hat{E}(\Delta^{(1)}(\lambda_1)) \quad \forall \tilde{\lambda}_1 \leq \lambda_1 \quad \text{and} \quad \hat{E}(\Delta^{(1)}(q_m^{(1)})) = \emptyset, \quad \hat{E}(\Delta^{(1)}(q_M^{(1)})) = \mathbf{1}. \quad (49)$$

Of course, these conditions are not necessarily satisfied in any $V_{1,3}$ and Σ but on our heuristic level we can assume that the particle is localised essentially in a region where they take place.

Then, one can introduce an operator $\hat{q}^{(1)}(x)$ corresponding to measurement of function $q^{(1)}(x)$ as follows. At first, define all its diagonal matrix elements as integrals

$$\langle \psi | \hat{q}^{(1)}(.) | \psi \rangle \stackrel{def}{=} \int_{q_m^{(1)}}^{q_M^{(1)}} q^{(1)}(x) d\mu_\psi(\Delta^{(1)}) \quad \forall |\psi\rangle \in \mathcal{H} \quad (50)$$

over a numeric measure

$$\mu_\psi(\Delta^{(1)}) \stackrel{def}{=} \langle \psi | \hat{E}(\Delta^{(1)}) | \psi \rangle \quad (51)$$

related to $|\psi\rangle$, which is generated by the pojection valued measure (48) on each $q^{(1)}$ - coordinate curve. Then, the nondiagonal matrix elements are known through the polarization formula. Namely, any real quadratic form $A(\psi, \psi)$ defines a sesquilinear functional $A'(\psi_1, \psi_2)$, see, e.g., [1, p.33]

$$A'(\psi_1, \psi_2) = A(\psi_1 + \psi_2) - A(\psi_1 - \psi_2) + iA(\psi_1 - i\psi_2) - iA(\psi_1 + i\psi_2) \quad (52)$$

Again, in $L^2(t; \mathbf{C}; d\sigma_t)$, this construction has the canonical realization of the measure $\mu_\psi(\Delta^{(1)})$ through the characteristic function

$$\chi(\Delta^{(1)}) = \begin{cases} 1 & \text{for } q^{(1)}(x) \in \Delta^{(1)}, \\ 0 & \text{for } q^{(1)}(x) \notin \Delta^{(1)}. \end{cases}$$

so that the integral Eq.(50) over this measure reads

$$\begin{aligned} (\psi, \hat{q}^{(1)}(.) \psi)_t &= \int_{q_m^{(1)}}^{q_M^{(1)}} q^{(1)}(x) d \int_{q_m^{(1)}}^{q_M^{(1)}} \int_{q_m^{(2)}}^{q_M^{(2)}} \int_{q_m^{(3)}}^{q_M^{(3)}} dq^1 dq^2 dq^3 \sqrt{\omega_t} \chi(\Delta^{(1)}) \bar{\psi} \psi, \\ &= \int_{q_m^{(1)}}^{q_M^{(1)}} \int_{q_m^{(2)}}^{q_M^{(2)}} \int_{q_m^{(3)}}^{q_M^{(3)}} dq^1 dq^2 dq^3 \sqrt{\omega_t} \bar{\psi} q^{(1)}(x) \psi \quad \text{for any } \psi \in L^2(t; \mathbf{C}; d\sigma_t) \end{aligned} \quad (53)$$

Hence it is obvious that, up to possible topological complications noted above, Eq.(53) defines the operator $\hat{q}^{(1)}(x)$ just as the multiplication operator $\hat{q}^{\mathbf{c}(i)}(x)$, Eq.(30). Since the function $q^{(1)}(x)$ was chosen arbitrarily, the same is true for any $\hat{q}^{\mathbf{c}}(x)$.

I should emphasize again that this reasoning is not a mathematically rigor justification of definition of the position operators but is presented here for notification of possible (and necessary) development of the theory, which seems also to be very interesting in the nontrivial geometric context.

7 Conclusion

Thus, we introduced a geodesic congruence (frame of reference) which is normal to a given Cauchy hypersurface Σ and, by the simplest canonical procedure on this basis, formulated basic notions and relations of quantum mechanics which corresponds to geodesic motion in the general $V_{1,3}$. The resulting structure is similar to Schrödinger picture of the standard nonrelativistic quantum mechanics with the Born probabilistic interpretation of the wave functions.

A specific feature of the structure is that there are two types of coordinates. The first one consists of $\{x^\alpha\}$, arbitrary coordinates providing $V_{1,3}$ with a manifold structure. The second type of coordinates are $\{q^{(i)}(x)\}$ through which a space localization of a particle is determined and the values of which are assumed to be outputs of specific devices, each corresponding to some choice of a function $q^{(i)}(x)$. These coordinates are subjected to quantization and are mutually commuting quantum-mechanical operators. An attempt to generalize the position operators to $V_{1,3}$ is done though it retains open questions of topological character which seem to be very interesting for further study.

At the same time, the theory is relativistic in the sense that it is a quantum counterpart to a classical relativistic dynamics and includes c as a parameter. If one takes $\hat{p}_{(0)}^{\mathbf{c}}(x) = mc^2 + \hat{H}^{\mathbf{c}}(x)$ then

$$c^{-2} \left(\hat{p}_{(0)}^{\mathbf{c}} \right)^2 - \hat{p}_{(i)}^{\mathbf{c}} \omega^{ij}(t, q) \hat{p}_{(j)}^{\mathbf{c}} = m^2 c^2 \cdot \mathbf{1}, \quad (54)$$

that is just the relativistic relation for the four-momentum of a particle of the mass m . One sees also that the Hamilton operator $H^{\mathbf{c}}$ is nonlocal. Computationally, it can be considered asymptotically in orders of $(mc^2)^{-1}$ and then the zero order (nonrelativistic) term is, up to a choice of the quantum potential $V(x)$, just the Hamiltonian which is introduced by a definition in quantum mechanics on symmetric manifolds, see e.g. [10] where numerous references can be found. Now the Hamilton operator is obtained by quantization of a natural dynamics and one has a way to develop study of symmetries to relativistic corrections if it is

interesting. Study of quantum potential in relation with the problem of ordering of operators $\hat{q}^{(i)}$, $p_{(j)}$ seems also to be among the important questions arising in concern with the present work. However, I should remind that my first aim is to compare the present canonical version of quantum mechanics with that arising from the quantum field theory in $V_{1,3}$. It will be presented in a subsequent paper II.

The author appreciates helpful discussions with Dr.D.Mladenov and Dr. P.E. Zhidkov.

References

- [1] Jauch J.M. (1968), *Foundations of Quantum Mechanics*, Addison–Wesley, Reading, Massachusetts.
- [2] Śniatycki J., (1980), *Geometric Quantization and Quantum Mechanics*, Springer–Verlag, New York, Heidelberg, Berlin.
- [3] Dirac P.A.M. (1958), *The Principles of Quantum Mechanics*, 4th ed., Clarendon Press, Oxford.
- [4] Kálnay A.J. (1981), *Hadronic J.* **4**, 1127–1165.
- [5] Chernikov N.A., Tagirov E.A. (1969), *Annales de l'Institut Henry Poincaré*, **A9**, 39.
- [6] Tagirov E.A. (1973), *Annals of Physics (N.Y.)*, **76**, 561.
- [7] Tagirov E.A. (1998), gr–qc/9807030.
- [8] Varadarajan V.S. (1970), *Geometry of Quantum Theory*, Vol.II, Van Nostrand Reinhold Co, N.Y.
- [9] Maslov V.P., Fedoriuk M.V. (1976), *Quasiclassical Approximation for Equations of Quantum Mechanics* (in Russian), Nauka, Moscow.
- [10] Grosche C., Pogosyan, G.S., Sissakian A.N. (1997), *Particles and Nuclei (Dubna)* **28**, N 5, 1229 –1294.